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New conditional symmetries and exact solutions of nonlinear reaction–diffusion–convection equations

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Abstract

A complete description of Q -conditional symmetries for two classes of reaction–diffusion–convection equations with power diffusivities is derived. It is shown that all the known results for reaction–diffusion equations with power diffusivities follow as particular cases from those obtained here but not vice versa. The symmetries obtained for constructing exact solutions of the relevant equations are successfully applied. In the particular case, new exact solutions of nonlinear reaction–diffusion–convection equations arising in application and their natural generalizations are found.

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1. Introduction

Nonlinear reaction–diffusion–convection (RDC) equations of the form

$$U_t = [A(U)U_x]_x + B(U)U_x + C(U), \quad (1)$$

where $U = U(t, x)$ is the unknown function, $A(U)$, $B(U)$, $C(U)$ are the given smooth functions and the subscripts t and x denote differentiation with respect to these variables, arise in a wide range of mathematical models describing various processes in physics and biology [1–3]. Starting from the remarkable Ovsiannikov work [4] a great number of papers devoted to investigation of these equations by means of group-theoretical methods. At the present time one can claim that all Lie symmetries of (1) are completely described and the relevant Lie solutions are constructed for many equations of the form (1), which arise in applications (see [5]–[10] and the papers cited therein).

In 1969, Bluman and Cole [11] introduced an essential generalization of Lie symmetry using the simplest representative of (1), the linear heat equation. These generalized symmetries are often called nonclassical symmetries nevertheless this notion was not used in [11]. It should be noted that the determining equations for searching nonclassical symmetries of the linear

heat equation (see system (21) with $\lambda = 0$) were not solved in [11]. Many authors tried to build the general solution of those equations [14, 15, 17]. The most general results were obtained in [18–20]. In the papers [18, 19], it was proved that the general solution is expressed in terms of three solutions of the linear heat equation, while the authors of [20] have shown how the general solution is also obtainable via the matrix Cole–Hopf transformation. In [19, 20], the determining equations for searching nonclassical symmetries of the Burgers equation were also solved.

The notion of nonclassical symmetry was further developed in [12, 13, 15, 16] and many others papers (see an extensive overview in [21]). A new generalization of Lie symmetry, conditional symmetry, was suggested by Fushchych and his collaborators [22] and [23, section 5.7]. Note that the notion of nonclassical symmetry can be derived as a particular case from conditional symmetry but not vice versa (see, e.g., an example in [24]). In the middle of 1990s of the last century the notion of generalized conditional symmetry was introduced [25, 26], which again can be considered as a special case of conditional symmetry. Taking this into account, to avoid any misunderstanding we continuously use the terminology Q -conditional symmetry [23] instead of nonclassical symmetry. In fact, there are several types of non-Lie symmetries at the present time and each of them can be called nonclassical one.

While there is no existing general theory for integrating the nonlinear RDC of the form (1), the construction of particular exact solutions for these equations is a non-trivial and important problem. Finding exact solutions that have a physical, chemical or biological interpretation is of fundamental importance. It is well known that the notion of Q -conditional symmetry plays an important role in investigation of the nonlinear RDC equations since, having such symmetries in the explicit form, one may construct new exact solutions, which are not obtainable by the classical Lie machinery. Several papers were devoted to this topic during the last 15 years [9, 23, 27–32]. The time is therefore ripe for a complete description of non-Lie symmetries for the general RDC equation (1). Since it seems to be an extremely difficult task at the present time here we present the solving for some important particular cases of (1), namely,

$$U_t = [U^m U_x]_x + \lambda U^m U_x + C(U), \quad (2)$$

$$U_t = [U^m U_x]_x + \lambda U^{m+1} U_x + C(U), \quad (3)$$

where λ and m are the arbitrary constants while $C(U)$ is an arbitrary functions.

It should be noted that Q -conditional symmetry of (2) with $\lambda = 0$, $m = 0$, i.e. reaction–diffusion equation, was investigated in [23, 27], the most general results were obtained in [29, 33]. Operators of Q -conditional symmetry for equation (3) with $\lambda = 0$, $m \neq 0$ have been constructed in [30] while the complete description of those for the RDC (3) with $m = 0$ is presented in the recently published paper [32]. Finally, we remind the reader that the determining equations for constructing the Q -conditional symmetry operators of the general RDC equation (1) were obtained in [9]; however, that paper contains only examples of particular solutions of those equations.

We stress that only reaction–diffusion equations with the convective terms ($\lambda \neq 0$) are considered below. The motivation of this restriction has two aspects. The first one is to find Q -conditional symmetries for nonlinear equations involving three transport mechanisms (diffusion, reaction and convection) in contrary to standard reaction–diffusion (RD) equations. The second one is to deal with the equations which arise in applications. In fact, sometimes the convection arises as a natural extension of a conservation law and then one obtains the RDC equations instead of the RD equations. The effect of nonlinear convection in the RD equations can have ‘a dramatic effect on the solutions’ [2, section 11.4]).

The paper is organized as follows. In section 2, we present two theorems giving a complete description of Q -conditional symmetries of the nonlinear RDC equations (2) and (3). In section 3, the proof of the theorems are presented. In section 4, the Q -conditional symmetries are successfully applied for constructing a wide range of exact solutions of the nonlinear RDC equations including the Murray equation with the fast and slow diffusions and the Fitzhugh–Nagumo equation with the fast diffusion and convection. The main results of the paper are summarized and discussed in the last section.

2. Main theorems

We want to find all possible Q -conditional symmetries of the form

$$Q = \partial_t + \xi(t, x, U)\partial_x + \eta(t, x, U)\partial_U, \tag{4}$$

where ξ and η are the unknown functions, for the RDC equations (2) and (3). We do not consider the problem of constructing Q -conditional symmetries of the form

$$Q = \partial_x + \eta(t, x, U)\partial_U,$$

because that is equivalent (up to the known non-local transformation) to solving the given equations (2) and (3) [34].

Now we present main results of the paper in the form of two theorems. Note that we search for purely conditional symmetry operators, which cannot be reduced to the Lie symmetry operators described completely in [9, 10].

Theorem 1. *Equation (2) is Q -conditional invariant under the operator (4) if and only if it and the relevant operator (up to equivalent representations generated by multiplying on the arbitrary smooth function $M(t, x, U)$) have the following forms:*

$$(i) \quad U_t = [U^m U_x]_x + \lambda U^m U_x + (\lambda_1 U^{m+1} + \lambda_2)(U^{-m} - \lambda_3), \quad m \neq -1, \lambda_2 \neq 0 \tag{5}$$

$$Q = \partial_t + (\lambda_1 U + \lambda_2 U^{-m})\partial_U; \tag{6}$$

$$(ii) \quad U_t = [U^{-1} U_x]_x + \lambda U^{-1} U_x + (\lambda_1 \ln U + \lambda_2)(U - \lambda_3), \quad \lambda_1 \neq 0, \tag{7}$$

$$Q = \partial_t + (\lambda_1 \ln U + \lambda_2)U\partial_U; \tag{8}$$

$$(iii) \quad U_t = [U^{-\frac{1}{2}} U_x]_x + \lambda U^{-\frac{1}{2}} U_x + \lambda_1 U + \lambda_2 U^{\frac{1}{2}} + \lambda_3, \tag{9}$$

$$Q = \partial_t + f(t, x)\partial_x + 2(g(t, x)U + h(t, x)U^{\frac{1}{2}})\partial_U, \tag{10}$$

where the function triplet (f, g, h) is the general solution of the system

$$\begin{aligned} 2ff_x + f_t + fg &= 0, \\ f_{xx} - \lambda f_x - 2g_x - fh &= 0, \\ \left(g - \frac{\lambda_1}{2}\right)(g + 2f_x) + g_t &= 0, \\ 2gh - \lambda_1 h + 2f_x h - \lambda_2 f_x + h_t - \lambda g_x - g_{xx} &= 0, \\ h^2 - \frac{\lambda_2}{2}h - \lambda_3 f_x + \frac{\lambda_3}{2}g - \lambda h_x - h_{xx} &= 0. \end{aligned} \tag{11}$$

Hereafter λ_1, λ_2 and λ_3 are arbitrary constants.

It should be noted that cases (i) and (ii) with $\lambda = 0$ immediately give the RD equations and the relevant symmetries obtained in [30].

Nevertheless, system (11) contains five equations on three unknown functions it is compatible. In fact, the system with $f = g = 0$, $\lambda_1 = 0$ is reduced to the ordinary differential equation

$$h_{xx} + \lambda h_x + \frac{\lambda_2}{2}h - h^2 = 0 \quad (12)$$

and hence

$$Q = \partial_t + 2h(x)U^{\frac{1}{2}}\partial_U \quad (13)$$

is the Q -conditional symmetry operator for an arbitrary non-zero solution of (12). Unfortunately, ODE (12) cannot be integrated for the arbitrary coefficients λ and λ_2 ; however, some particular solutions can be easily established. For example, setting $h = \frac{\lambda_2}{2}$, a particular case of (i) with $m = -\frac{1}{2}$, $\lambda_1 = 0$ is obtained.

Setting $\lambda = \lambda_2 = 0$ in (12), we arrive at the known ODE $h_{xx} = h^2$ with the general solution $h = \mathcal{W}(0, c_1, x + c_2)$, where c_1 and c_2 are the arbitrary constants, \mathcal{W} is the Weierstrass function with the periods 0 and c_1 . Its simplest solution takes the form $h = 6x^{-2}$ and leads to the known Q -conditional symmetry operator $Q = \partial_t + 12x^{-2}U^{\frac{1}{2}}\partial_U$ of the nonlinear diffusion equation $U_t = [U^{-\frac{1}{2}}U_x]_x$ [30]. However, the result derived in [30] can be generalized as follows. One can easily check that an arbitrary particular solution of (11) with $\lambda = 0$ generates the Q -conditional symmetry operator (10) of the RD equation

$$U_t = [U^{-\frac{1}{2}}U_x]_x + \lambda_1 U + \lambda_2 U^{\frac{1}{2}} + \lambda_3. \quad (14)$$

Obviously, the operator presented in table 3 of [30] is obtainable from (10) and (11) by setting $\lambda = \lambda_1 = 0$ and $f = g = 0$ but not vice versa.

Theorem 2. Equation (3) is Q -conditional invariant under the operator (4) if and only if it and the relevant operator (up to equivalent representations generated by multiplying on the arbitrary smooth function $M(t, x, U)$) have the following forms:

$$(i) \quad U_t = [U^m U_x]_x + \lambda U^{m+1} U_x + \lambda_1 U + \lambda_2 U^{-m}, \quad m \neq -1, \quad (15)$$

$$Q = \partial_t - \lambda U^{m+1} \partial_x + (\lambda_1 U + \lambda_2 U^{-m}) \partial_U; \quad (16)$$

$$(ii) \quad U_t = [U^{-\frac{1}{2}} U_x]_x + \lambda U^{\frac{1}{2}} U_x + (\lambda_1 U^{\frac{3}{2}} + \lambda_2 U^{\frac{1}{2}} + \lambda_3) \left(\frac{\lambda_1}{2\lambda^2} + U^{\frac{1}{2}} \right), \quad (17)$$

$$Q = \partial_t + \left(-\lambda U^{\frac{1}{2}} + \frac{3\lambda_1}{2\lambda} \right) \partial_x + (\lambda_1 U^{\frac{3}{2}} + \lambda_2 U^{\frac{1}{2}} + \lambda_3) \partial_U; \quad (18)$$

$$(iii) \quad U_t = U_{xx} + \lambda U U_x, \quad (19)$$

$$Q = \partial_t + \left(\frac{\lambda}{2} U + q \right) \partial_x + \left(a + bU - \frac{\lambda q}{2} U^2 - \frac{\lambda^2}{4} U^3 \right) \partial_U; \quad (20)$$

where the triplet of the functions (a, b, q) is the general solution of the system

$$a_t = a_{xx} - 2aq_x, \quad b_t = b_{xx} - 2bq_x + \lambda a_x, \quad q_t = q_{xx} - 2qq_x - 2b_x; \quad (21)$$

$$(iv) \quad U_t = U_{xx} + \lambda U U_x + \lambda_0 + \lambda_2 U^2, \quad \lambda_2 \neq 0, \quad (22)$$

$$Q = \partial_t + \left(-\lambda U + \frac{\lambda_2}{\lambda} \right) \partial_x + (\lambda_0 + \lambda_2 U^2) \partial_U; \quad (23)$$

$$(v) \quad U_t = U_{xx} + \lambda U U_x + \lambda_0 + \lambda_1 U + \lambda_3 U^3, \quad \lambda_3 \neq 0, \quad (24)$$

$$Q_i = \partial_t + p_i U \partial_x + \frac{3p_i}{2p_i - \lambda} (\lambda_0 + \lambda_1 U + \lambda_3 U^3) \partial_U, \quad i = 1, 2; \quad (25)$$

where p_i are the roots of the quadratic equation $2p^2 + \lambda p + 9\lambda_3 - \lambda^2 = 0$ and $\lambda_0 \in \mathbb{R}$.

Remark 1. The case of the Burgers equation (19) was completely investigated in the papers [19, 20, 35] (see also a locally equivalent equation to the Burgers equation found in [9]) while cases (iv) and (v) were obtained in [32].

Remark 2. The RDC equations

$$U_t = U_{xx} + \lambda U U_x + \lambda_0 + \lambda_1 U + \lambda_2 U^2$$

and

$$U_t = U_{xx} + \lambda U U_x + \lambda_0 + \lambda_1 U + \lambda_2 U^2 + \lambda_3 U^3$$

possess Q -conditional symmetry; however, they are reduced to (22) and (24), respectively, by the simple local substitutions [32].

Inserting $\lambda = 0$ into (24) and (25) one immediately arrives at the well-known RD equation and Q -conditional symmetry operator with cubic nonlinearities constructed earlier [27]–[29].

Consider some equations which arise as the particular cases of those from theorems 1 and 2 and are known in application. Equation (5) with $m = 1$ contains as a subcase the equation

$$U_t = [U U_x]_x + \lambda U U_x + \lambda_1 U(1 - U), \quad (26)$$

which is a natural generalization of the equation

$$U_t = U_{xx} + \lambda U U_x + \lambda_1 U(1 - U), \quad (27)$$

extensively studied by Murray [2, section 11.4]. So, equation (26) may be called the Murray equation with the slow diffusion. On the other hand, (26) is nothing else but the porous-Fisher equation with the Burgers convective term $\lambda U U_x$. Note that the Murray equation (27) arises in theorem 2 (see case (iv) and remark 2). Its Q -conditional symmetry was established earlier in [9] and several exact solutions were recently found in [32]. The Murray equation with the fast diffusion can also be obtained from (5). Indeed, setting $m = -2$, $\lambda_3 = 0$ and $\lambda_2 = -\lambda_1$, we arrive at the equation

$$U_t = [U^{-2} U_x]_x + \lambda U^{-2} U_x + \lambda_1 U(1 - U). \quad (28)$$

It should be noted that equation (28) with $\lambda_1 = 0$ is linearizable by the known integral substitution [36] while that with $\lambda_1 \neq 0$ is not linearizable.

It is interesting to note that equation (9) with $\lambda_2 = -\lambda_1$ and $\lambda_3 = 0$ takes the form

$$U_t = [U^{-\frac{1}{2}} U_x]_x + \lambda U^{-\frac{1}{2}} U_x + \lambda_1 U^{\frac{1}{2}} (1 - U^{\frac{1}{2}}). \quad (29)$$

This equation is an analog of the Murray equation with the fast diffusion. Another analog of the Murray equation with the fast diffusion is

$$U_t = [U^{-\frac{1}{2}} U_x]_x + \lambda U^{\frac{1}{2}} U_x + \lambda_1 U^{\frac{1}{2}} (1 - U^{\frac{1}{2}}), \quad (30)$$

which is a particular case of equation (15).

Consider the generalized Fitzhugh–Nagumo (FN) equation

$$U_t = U_{xx} + \lambda U U_x + \lambda_3 U(U - \delta)(1 - U), \quad 0 < \delta < 1. \quad (31)$$

We remind the reader that (31) with $\lambda = 0$ is the famous FN equation [37] describing nerve impulse propagation. It can also be considered as a simplification of the Hodgkin–Huxley model (see, e.g., [3]) describing the ionic current flows for axonal membranes. Equation (31)

with $\lambda = 0$ and $\delta = 1$ is the Kolmogorov–Petrovskii–Piskunov equation, which first was investigated in [38] (see English translation in [39]) to describe the population dynamics under some restrictions on characteristic individuals. Equation (31) can be reduced to the form

$$W_t = W_{yy} + \lambda W W_y + \lambda_0 + \lambda_1 W - \lambda_3 W^3, \quad (32)$$

where

$$\lambda_1 = \lambda_3 \left(\frac{1}{3}(\delta + 1)^2 - \delta \right), \quad \lambda_0 = \lambda_3 \frac{1}{3}(\delta + 1) \left(\frac{2}{9}(\delta + 1)^2 - \delta \right), \quad (33)$$

by the local substitution

$$W(t, y) = U - \frac{1}{3}(\delta + 1), \quad y = x + \frac{\lambda}{3}(\delta + 1)t. \quad (34)$$

Now one note that equation (32) is nothing else but equation (24) with the new notation. On the other hand, (17) contains as a particular case the equation

$$U_t = (U^{-\frac{1}{2}} U_x)_x + \lambda U^{\frac{1}{2}} U_x + \lambda_2 U^{\frac{1}{2}} (U^{\frac{1}{2}} - \delta)(1 - U), \quad \delta = \frac{\lambda_2}{2\lambda^2}, \quad (35)$$

which may be treated as a generalized FN equation with the fast diffusion. Moreover it will be shown in section 4 that equation (35) possesses exact solutions, which have similar structure to those for the generalized FN equation (31). In the case $\lambda_2 = 2\lambda^2$, this equation may be called the generalized Kolmogorov–Petrovskii–Piskunov equation with the fast diffusion.

3. Proofs of the theorems

Proof of theorem 1. The proof of theorems 1 and 2 is based on the known algorithm for finding the Q -conditional symmetry operators (see, e.g., [9, 23]). First, we apply the local substitution

$$V = \begin{cases} U^{m+1}, & m \neq -1, \\ \ln U, & m = -1. \end{cases} \quad (36)$$

In the cases $m \neq -1$ and $m = -1$, substitution (36) reduces equation (2) to the forms

$$V_{xx} = V^n V_t - \lambda V_x + F(V), \quad (37)$$

(here $n = -\frac{m}{m+1} \neq 0$, $F(V) = -(m+1)C(V^{\frac{1}{m+1}})$, $\lambda \neq 0$) and

$$V_{xx} = \exp(V) V_t - \lambda V_x + F(V), \quad F(V) = C(\exp V), \quad (38)$$

respectively. \square

The determining equations for the general RDC equation

$$V_{xx} = F_0(V) V_t + F_1(V) V_x + F_2(V),$$

with $F_i(V)$, $i = 1, 2, 3$, being the arbitrary functions, have been obtained in [9] (see P.535). In the case $F_0(V) = V^n$, $F_1(V) = -\lambda$ and $F_2(V) = F(V)$, those equations take the form

$$\begin{aligned} \xi_{VV} &= 0, \\ \eta_{VV} &= 2\xi_V(-\lambda - \xi V^n) + 2\xi_{xV}, \\ (2\xi_V \eta - 2\xi \xi_x - \xi_t) V^n - \xi \eta n V^{n-1} - \lambda \xi_x + 3\xi_V F - 2\eta_{xV} + \xi_{xx} &= 0, \\ \eta F_V + (2\xi_x - \eta_V) F + n\eta^2 V^{n-1} + 2\xi_x \eta V^n + \eta_t V^n - \lambda \eta_x - \eta_{xx} &= 0. \end{aligned} \quad (39)$$

Solving the first equation of (39), we arrive at the function $\xi = a(t, x)V + f(t, x)$ with $a(t, x)$ and $f(t, x)$ being the arbitrary smooth functions at the moment.

It turns out that system (39) does not possess any Q -conditional symmetry if $a(t, x) \neq 0$. So we must assume

$$\xi = f(t, x). \tag{40}$$

Solving the second equation of (39) under condition (40), we arrive at

$$\eta = g(t, x)V + h(t, x). \tag{41}$$

Taking into account (40) and (41), the third equation of (39) reduces to the form

$$(2ff_x + f_t + nfg)V^n + nfhV^{n-1} - f_{xx} + \lambda f_x + 2g_x = 0. \tag{42}$$

This equation can be split with respect to the powers of V . One needs to consider two cases depending on n :

(a) if $n \neq 1$ then

$$2ff_x + f_t + nfg = 0, \quad fh = 0, \quad f_{xx} - \lambda f_x - 2g_x = 0. \tag{43}$$

(b) if $n = 1$ then

$$2ff_x + f_t + fg = 0, \quad f_{xx} - \lambda f_x - 2g_x - fh = 0.$$

Let us consider case (a). Substituting (40) and (41) into the fourth equation of (39), one arrives at

$$(gV + h)F_V + (2f_x - g)F = -nV^{n-1}(gV + h)^2 + h_{xx} + \lambda h_x + (g_{xx} + \lambda g_x)V - (g_t + 2f_x g)V^{n+1} - (h_t + 2f_x h)V^n. \tag{44}$$

To solve (44) and (43) one needs to consider two subcases, which follow from the second equation of (43), i.e. either $f = 0$ or $h = 0$.

The case $f = 0$ leads to the system

$$\begin{aligned} f &= 0, \\ g_x &= 0, \\ (gV + h)F_V - gF &= -nV^{n-1}(gV + h)^2 + h_{xx} + \lambda h_x - g_t V^{n+1} - h_t V^n. \end{aligned} \tag{45}$$

Setting $g = \text{const}$, $h = \text{const}$, we arrive at the system

$$f = 0, \quad g = \lambda_1^*, \quad h = \lambda_2^*, \quad F = (\lambda_1^* V + \lambda_2^*)(\lambda_3 - V^n),$$

therefore

$$V_{xx} = V^n V_t - \lambda V_x + (\lambda_1^* V + \lambda_2^*)(\lambda_3 - V^n), \tag{46}$$

$$Q = \partial_t + (\lambda_1^* V + \lambda_2^*)\partial_V. \tag{47}$$

Applying substitution (36) to equation (46) and operator (47) we obtain case (i) of the theorem (note one should use new notations $\lambda_i = \frac{\lambda_i^*}{m+1}$, $i = 1, 2$).

Now we assume that $g \neq \text{const}$, so that the third equation of (45) can be reduced to the form

$$\left(V + \frac{h}{g}\right)F_V - F = -ngV^{n-1}\left(V + \frac{h}{g}\right)^2 + \frac{h_{xx} + \lambda h_x}{g} - \frac{g_t}{g}V^{n+1} - \frac{h_t}{g}V^n. \tag{48}$$

It turns out that the last equation can be satisfied only under condition $\frac{h}{g} = \text{const}$ (see the proof below). Setting $\frac{h}{g} = \text{const}$ into (48) and making the relevant calculations, we only obtain the Lie symmetry operators and a particular case of operator (47) and equation (46). For example, if $\frac{h}{g} = 0$ then the system

$$\begin{aligned} h &= 0, \quad 2ff_x + f_t + nfg = 0, \quad f_{xx} - \lambda f_x - 2g_x = 0, \\ gVF_V + (2f_x - g)F &= -nV^{n+1}g^2 + (g_{xx} + \lambda g_x)V - (g_t + 2f_x g)V^{n+1} \end{aligned} \tag{49}$$

is obtained, which can be easily solved and its general solution has the form

$$f = \frac{c_1 \exp(\lambda_1 nt)}{c_2 \exp(\lambda_1 nt) + 1}, \quad g = -\frac{\lambda_1}{c_2 \exp(\lambda_1 nt) + 1},$$

$$h = 0, \quad F = \lambda_1 V^{n+1} + \lambda_2 V,$$

where $c_k \in \mathbb{R}$, $k = 1, 2$. Hence, we arrive at the RDC equation

$$V_{xx} = V^n V_t - \lambda V_x + \lambda_1 V^{n+1} + \lambda_2 V$$

and the operator

$$Q = \partial_t + \frac{c_1 \exp(\lambda_1 nt)}{c_2 \exp(\lambda_1 nt) + 1} \partial_x - \frac{\lambda_1 V}{c_2 \exp(\lambda_1 nt) + 1} \partial_V. \quad (50)$$

However, one can establish by multiplying (50) on the function $M(t, x, U) = c_2 \exp(\lambda_1 nt) + 1$ that the last operator is nothing else but the Lie symmetry operator (see case 8 of table 1 in [9]).

Let us prove that $\frac{h}{g} = \text{const}$. By differentiating equation (48) with respect to the variables x and t one obtains two equations. Assuming $(\frac{h}{g})_t (\frac{h}{g})_x = 0$, one easily arrives at the condition $\frac{h}{g} = \text{const}$.

Consider the case $(\frac{h}{g})_t (\frac{h}{g})_x \neq 0$. By differentiating equation (48) with respect to the variables x we arrive at the equation

$$F_V = -\frac{1}{h_x} (2ngh + h_t)_x V^n - 2nh V^{n-1} + \frac{h_{xxx} + \lambda h_{xx}}{h_x}.$$

Since the functions V^n , V^{n-1} and 1 on the right-hand side are functionally independent (we consider the case $n \neq 1$) their coefficients must be constants. It means that $2nh = \text{const}$ so that $h_x = 0$. Taking now two last equations of (43), one easily establish that $g_x = 0$, i.e. we arrive at the contradiction: $(\frac{h}{g})_x = 0$.

Consider case (b). Substituting (40) and (41) into the fourth equation of (39), we arrive at (44) with $n = 1$. Dealing with this equation in the same way as above (see case (a)) we obtain the equation

$$V_{xx} = V V_t - \lambda V_x + \lambda_1^* V^2 + \lambda_2^* V + \lambda_3^* \quad (51)$$

and the operator

$$Q = \partial_t + f(t, x) \partial_x + (g(t, x) V + h(t, x)) \partial_V, \quad (52)$$

where the triplet (f, g, h) are the general solution of (11). Applying formula (36) with $m \neq -1$ we obtain case (iii) of the theorem (note one should use new notations $\lambda_i = -2\lambda_i^*$, $i = 1, 2, 3$).

Finally, we analyze equation (38), which is locally equivalent to the RDC (2) with $m = -1$. Using again the determining equations, which have been obtained in [9] to find operators of the Q -conditional symmetries (6), we arrive at the following system:

$$\begin{aligned} \xi_{VV} &= 0, \\ \eta_{VV} &= 2\xi_V(-\lambda - \xi \exp V) + 2\xi_{xV}, \\ (\xi_t + 2\xi \xi_x - 2\xi_V \eta + \xi \eta) \exp V + \lambda \xi_x - 3\xi_V F + 2\eta_{xV} - \xi_{xx} &= 0, \\ \eta F_V + (2\xi_x - \eta_V) F + (\eta^2 + 2\xi_x \eta + \eta_t) \exp V - \lambda \eta_x - \eta_{xx} &= 0, \end{aligned} \quad (53)$$

where ξ , η and F are yet-to-be determined functions. Solving the first and second equations of this system we establish that the functions ξ and η must be given by formulae (40) and

(41), respectively, otherwise Q -conditional symmetry does not exist. Substituting (40) and (41) into the third equation of (53) we obtain the equation

$$(fh + f_t + 2ff_x) \exp V + (fg)V \exp V + \lambda f_x + 2g_x - f_{xx} = 0.$$

Since the functions f, g and h do not depend on V , one can split this equation with respect to $\exp V$ and $V \exp V$ and obtain the system

$$fh + f_t + 2ff_x = 0, \quad fg = 0, \quad \lambda f_x + 2g_x - f_{xx} = 0. \tag{54}$$

Substituting (40) and (41) into the fourth equation of (53) we arrive at the equation

$$(gV + h)F_V + (2f_x - g)F = -(gV + h)^2 \exp V + h_{xx} + \lambda h_x + (g_{xx} + \lambda g_x)V - (g_t + 2f_x g)V \exp V - (h_t + 2f_x h) \exp V. \tag{55}$$

Now we apply to (55) the same approach, which has been used for solving equation (44). Thus, taking into account system (54) we finally obtain the expressions

$$F = (\lambda_1 V + \lambda_2)(\lambda_3 - \exp V), \quad f = 0, \quad g = \lambda_1, \quad h = \lambda_2,$$

which lead to the equation

$$V_{xx} = \exp(V)V_t - \lambda V_x + (\lambda_1 V + \lambda_2)(\lambda_3 - \exp V) \tag{56}$$

and the operator

$$Q = \partial_t + (\lambda_1 V + \lambda_2)\partial_V. \tag{57}$$

Applying substitution (36) with $m = -1$ to (56) and (57) one obtains case (ii) of theorem 1.

The proof is now completed.

Sketch of the proof of theorem 2. First of all we note that all Q -conditional symmetries of equation (3) with $m = 0$ (see cases (iii), (iv) and (v) in theorem 2) were found in the recent paper [32]. Note we were able to solve the overdetermined system (18) [32] in the general case and to establish that all solutions of this system produce only the Lie symmetry operators so that the relevant case was eliminated. \square

Hereafter we assume the restriction $m \neq 0$. Let us again use the substitution (36), which reduces equation (3) to the form

$$V_{xx} = V^n V_t - \lambda V^{n+1} V_x + F(V) \tag{58}$$

(here $n = -\frac{m}{m+1} \neq 0, -1, F(V) = -(m+1)C(V^{\frac{1}{m+1}})$) if $m \neq -1$ and to the form

$$V_{xx} = \exp(V)V_t - \lambda \exp(V)V_x + F(V), \quad F(V) = C(\exp V) \tag{59}$$

if $m = -1$.

Consider equation (58). Using the general form of the determining equations obtained in [9] one easily arrives at the following system:

$$\begin{aligned} \xi_{VV} &= 0, \\ \eta_{VV} &= 2\xi_V(-\lambda V^{n+1} - \xi V^n) + 2\xi_{xV}, \\ \eta F_V + (2\xi_x - \eta_V)F + n\eta^2 V^{n-1} + 2\xi_x \eta V^n + \eta_t V^n - \lambda V^{n+1} \eta_x - \eta_{xx} &= 0, \\ \lambda \xi_x V^{n+1} + ((-2\xi_V + \lambda(n+1))\eta + 2\xi \xi_x + \xi_t)V^n + \xi \eta V^{n-1} - 3\xi_V F + 2\eta_{xV} - \xi_{xx} &= 0 \end{aligned} \tag{60}$$

to find the function ξ, η and F . In the case of equation (59) that system takes the form

$$\begin{aligned} \xi_{VV} &= 0, \\ \eta_{VV} &= -2\xi_V(\lambda + \xi) \exp V + 2\xi_{xV}, \\ (\xi_t + 2\xi \xi_x + (\lambda + \xi - 2\xi_V)\eta + \lambda \xi_x) \exp V - 3\xi_V F + 2\eta_{xV} - \xi_{xx} &= 0, \\ \eta F_V + (2\xi_x - \eta_V)F + (\eta^2 + 2\xi_x \eta + \eta_t - \lambda \eta_x) \exp V - \eta_{xx} &= 0. \end{aligned} \tag{61}$$

Taking into account the first equations in (60) and (61), one establish that there are only three possibilities for the functions ξ and η :

$$\begin{aligned} \text{(a)} \quad & \xi = \lambda_1^* V + \lambda_2^*, \eta = \eta(V), \quad \lambda_1^*, \lambda_2^* \in \mathbb{R}, \\ \text{(b)} \quad & \xi = f(t, x), \quad \eta = g(t, x)V + h(t, x), \\ \text{(c)} \quad & \xi = a(t, x)V + f(t, x), \quad \eta = \eta(t, x, V), \quad a(t, x) \neq 0. \end{aligned} \quad (62)$$

In case (c) function $\eta(t, x, V)$ takes the forms

$$\eta = \begin{cases} -\frac{2a(a+\lambda)}{(n+2)(n+3)}V^{n+3} - \frac{2af}{(n+1)(n+2)}V^{n+2} + a_x V^2 + g(t, x)V + h(t, x), & n \neq -2, -3, \\ -2a(a+\lambda)V \ln V + 2af \ln V + a_x V^2 + (2a(a+\lambda) + g(t, x))V + h(t, x), & n = -2, \\ 2a(a+\lambda) \ln V - afV^{-1} + a_x V^2 + g(t, x)V + h(t, x), & n = -3, \end{cases} \quad (63)$$

for the system (60) and

$$\eta = -2a^2V \exp V - 2a(\lambda + f - 2a) \exp V + a_x V^2 + g(t, x)V + h(t, x)$$

for the system (61).

Considering case (a) and system (60) one exactly arrives at items (i) and (ii) of theorem 2 (see for the details [40]). On the other hand, it was proved that cases (b) and (c) only lead to the Lie symmetry operators.

The sketch of the proof is now completed.

4. Exact solutions of the nonlinear RDC equations

It is well known (see, e.g., examples in [41, 42]) that new non-Lie ansätze do not guarantee the construction of new exact solutions. It turns out the relevant exact solutions may also be obtainable by the standard Lie machinery if the given equation admits a non-trivial Lie symmetry. Here, we construct exact solutions using the Q -conditional symmetry operators found above and show that they are the so-called non-Lie solutions, i.e. cannot be obtained using the Lie symmetry operators. As it follows from the proofs presented in section 3, the Q -conditional symmetry operators have essentially simpler structure if one uses the substitution (36). So, we will first find exact solutions of equations (37) and (38) and (58) and (59) and afterwards use (36) to obtain those of the RDC equations (2) and (3).

We start from case (i) of theorem 1. Equation (5) and operator (6) are transformed by the substitution (36) to the forms

$$V_{xx} = V^n V_t - \lambda V_x + (\lambda_1^* V + \lambda_2^*)(\lambda_3 - V^n) \quad (64)$$

and

$$Q = \partial_t + (\lambda_1^* V + \lambda_2^*)\partial_V, \quad (65)$$

where $\lambda_i^* = \lambda_i(m+1)$, $i = 1, 2$. The relevant ansatz is constructed using the standard procedure, i.e. we solve the linear equation $Q(V) = 0$. Since its general solution depends on λ_1^* two ansätze are obtained:

$$V = \begin{cases} \lambda_2^* t + \varphi(x), & \lambda_1^* = 0, \\ \varphi(x) e^{\lambda_1^* t} - \frac{\lambda_2^*}{\lambda_1^*}, & \lambda_1^* \neq 0 \end{cases} \quad (66)$$

with $\varphi(x)$ being an unknown function. Substituting (66) with $\lambda_1^* = 0$ into (64), one arrives at the ordinary differential equation (ODE)

$$\varphi_{xx} + \lambda\varphi_x - \lambda_2^*\lambda_3 = 0,$$

with the general solution

$$\varphi = c_1 + c_2 e^{-\lambda x} + \frac{\lambda_2^*\lambda_3}{\lambda}x.$$

Hereafter c_1 and c_2 are arbitrary constants. Hence, equation (64) with $\lambda_1^* = 0$ possesses the exact solution

$$V = \lambda_2^*t + c_1 + c_2 e^{-\lambda x} + \frac{\lambda_2^*\lambda_3}{\lambda}x.$$

Using the substitution (36), we obtain the exact solution

$$U = \left[\lambda_2(m+1)t + c_1 + c_2 e^{-\lambda x} + \frac{\lambda_2\lambda_3(m+1)}{\lambda}x \right]^{\frac{1}{m+1}} \tag{67}$$

of the RDC equation with power nonlinearities

$$U_t = [U^m U_x]_x + \lambda U^m U_x + \lambda_2 U^{-m} - \lambda_2 \lambda_3, \quad m \neq -1. \tag{68}$$

Using the result of [9, 10] one establishes that equation (68) (with arbitrary coefficients) is invariant only under two-dimensional algebra with the basic operators ∂_t and ∂_x . So, $U = U(c_3x + c_4t)$, $c_3, c_4 \in \mathbb{R}$ is the most general form of solutions that are obtainable by the Lie machinery. Obviously, the exact solution presented above has different structures and cannot be reduced to this form, therefore it is a non-Lie solution. Note this solution is the Lie solution if one additionally sets $c_2 = 0$. In quite similar way it can be shown that all solutions obtained below are also the non-Lie solutions and may be reduced to the Lie solution only under additional constraints.

Substituting (66) with $\lambda_1^* \neq 0$ into (64), one again obtains a linear second-order ODE, which is integrable in terms of different elementary functions depending on $\delta = \lambda^2 + 4\lambda_1^*\lambda_3 = \lambda^2 + 4\lambda_1(m+1)\lambda_3$. Dealing in quite similar way to the case $\lambda_1^* = 0$, we finally obtain three exact solutions

$$U = \left[\exp\left(\lambda_1(m+1)t - \frac{\lambda}{2}x\right) \left(c_1 \exp\left(\frac{\sqrt{\delta}}{2}x\right) + c_2 \exp\left(-\frac{\sqrt{\delta}}{2}x\right) \right) - \frac{\lambda_2}{\lambda_1} \right]^{\frac{1}{m+1}}, \quad \delta > 0 \tag{69}$$

$$U = \left[\exp\left(-\frac{\lambda}{2}x + \lambda_1(m+1)t\right) (c_1 + c_2x) - \frac{\lambda_2}{\lambda_1} \right]^{\frac{1}{m+1}}, \quad \delta = 0 \tag{70}$$

$$U = \left[\exp\left(-\frac{\lambda}{2}x + \lambda_1(m+1)t\right) \left(c_1 \cos\frac{\sqrt{-\delta}}{2}x + c_2 \sin\frac{\sqrt{-\delta}}{2}x \right) - \frac{\lambda_2}{\lambda_1} \right]^{\frac{1}{m+1}} \quad \delta < 0 \tag{71}$$

of the nonlinear RDC equation

$$U_t = [U^m U_x]_x + \lambda U^m U_x + (\lambda_1 U^{m+1} + \lambda_2)(U^{-m} - \lambda_3), \quad m \neq -1. \tag{72}$$

In the case of the Murray equation with the slow diffusion (26) one notes that $\delta > 0$ if $\lambda_1 > 0$. Hence, solution (69) takes the form

$$U = \sqrt{\exp\left(2\lambda_1 t - \frac{\lambda}{2}x\right) \left(c_1 \exp\left(\frac{\sqrt{\delta}}{2}x\right) + c_2 \exp\left(-\frac{\sqrt{\delta}}{2}x\right) \right)}, \quad \delta = \lambda^2 + 8\lambda_1. \tag{73}$$

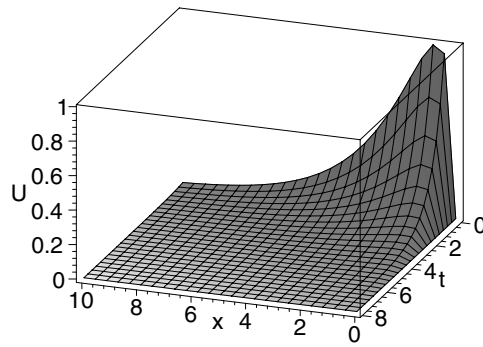


Figure 1. Exact solution (69) with $m = 1, \lambda = 3, \lambda_1 = -1, \lambda_2 = 0, c_1 = -c_2 = 4$.

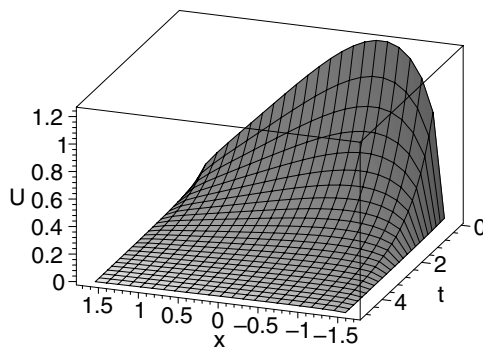


Figure 2. Exact solution (71) with $m = 1, \lambda = 2, \lambda_1 = -1, \lambda_2 = 0, c_1 = 1, c_2 = 0$.

This solution unboundedly grows if $t \rightarrow \infty$ or $x \rightarrow \pm\infty$. More interesting solutions occur in the case of (26) with the anti-logistic term:

$$U_t = [UU_x]_x + \lambda UU_x - U(1 - U). \quad (74)$$

Depending on $\delta = \lambda^2 - 8$ one obtains three types of solutions. In the case $m = 1, \lambda = 3, \lambda_1 = -1, \lambda_2 = 0, c_1 = -c_2 = 4$, solution (69) is presented in figure 1. This solution tends to zero if $t \rightarrow \infty$ and satisfies the zero boundary conditions for $x = 0$ and $x = \infty$. If $\lambda = 2$ then solution (71) with $m = 1$ is valid. In the case $\lambda_1 = -1, \lambda_2 = 0, c_1 = 1, c_2 = 0$ this solution is presented in figure 2. We note that the solution is again vanishing if $t \rightarrow \infty$, but one satisfies the zero Dirichlet conditions on the bounded interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Consider the Murray equation with the fast diffusion (28). Since $\delta = \lambda^2 > 0$ solution (69) takes the form

$$U = [1 + \exp(-\lambda_1 t)(c_1 + c_2 \exp(-\lambda x))]^{-1}, \quad (75)$$

which possesses attractive properties. Assuming $c_1 > 0$ and $c_2 > 0$, one sees that this solution is positive and bounded for arbitrary $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$. Moreover, the solution tends either to zero ($\lambda_1 < 0$) or to 1 ($\lambda_1 > 0$) if $t \rightarrow \infty$. Both values, $U = 0$ and $U = 1$, are the steady-state points of (28). Solution (75) tends to the steady-state point $U = 0$ if $\lambda x \rightarrow -\infty$, while $U = [1 + c_1 \exp(-\lambda_1 t)]^{-1}$ if $\lambda x \rightarrow \infty$. An example of solution (75) is presented in figure 3. It should also be noted that (75) with $c_1 = 0$ is a traveling wave solution with the same structure as that for the Murray equation (27) (see formula (90) in [32]).

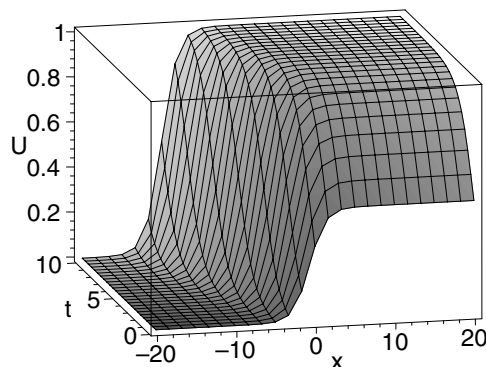


Figure 3. Exact solution (75) with $\lambda = 1, \lambda_1 = 1, c_1 = 1, c_2 = 1$.

Consider case (ii) of theorem 1. Equation (7) and operator (8) are transformed by the substitution (36) to the forms

$$V_{xx} = e^V V_t - \lambda V_x + (\lambda_1 V + \lambda_2)(\lambda_3 - e^V), \tag{76}$$

and

$$Q = \partial_t + (\lambda_1 V + \lambda_2)\partial_V, \tag{77}$$

respectively. Using operator (77) we obtain the ansatz

$$V = \begin{cases} \lambda_2 t + \varphi(x), & \lambda_1 = 0, \\ \varphi(x) e^{\lambda_1 t} - \frac{\lambda_2}{\lambda_1}, & \lambda_1 \neq 0, \end{cases} \tag{78}$$

which has the same structure as (66). Substituting (78) into (76), one again obtains integrable second-order ODEs and easily constructs the relevant exact solutions of the RDC equation (7). In the case $\lambda_1 = 0$, the solution is

$$U = \exp \left[\lambda_2 t + c_1 + c_2 e^{-\lambda x} + \frac{\lambda_2 \lambda_3}{\lambda} x \right], \tag{79}$$

while the case $\lambda_1 \neq 0$ produces three solutions depending on $\delta = \lambda^2 + 4\lambda_1 \lambda_3$:

$$U = \exp \left[\exp \left(\lambda_1 t - \frac{\lambda}{2} x \right) \left(c_1 \exp \left(\frac{\sqrt{\delta}}{2} x \right) + c_2 \exp \left(-\frac{\sqrt{\delta}}{2} x \right) \right) - \frac{\lambda_2}{\lambda_1} \right], \quad \delta > 0, \tag{80}$$

$$U = \exp \left[\exp \left(-\frac{\lambda}{2} x + \lambda_1 t \right) (c_1 + c_2 x) - \frac{\lambda_2}{\lambda_1} \right], \quad \delta = 0 \tag{81}$$

and

$$U = \exp \left[\exp \left(-\frac{\lambda}{2} x + \lambda_1 t \right) \left(c_1 \cos \frac{\sqrt{-\delta}}{2} x + c_2 \sin \frac{\sqrt{-\delta}}{2} x \right) - \frac{\lambda_2}{\lambda_1} \right], \quad \delta < 0. \tag{82}$$

Note that the properties of solutions (79)–(82) depend essentially on the values of c_1 and c_2 . For example, solution (80) with negative c_1 and c_2 tends to zero if $x \rightarrow \pm\infty$, while this solution infinitely increases if those constants are positive. An example of solution (80) is presented in figure 4.

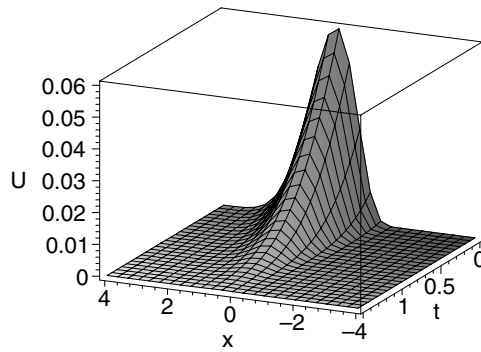


Figure 4. Exact solution (80) with $\lambda = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$, $c_1 = -1$, $c_2 = -1$.

Consider case (iii) of theorem 1. Since we were unable to solve the overdetermined system (11), we used the particular solution producing the Q -conditional operator (13). An application of this operator leads to a solution in the implicit form

$$U = (h(x)t + \varphi(x))^2 \quad (83)$$

to the nonlinear RDC equation

$$U_t = [U^{-\frac{1}{2}}U_x]_x + \lambda U^{-\frac{1}{2}}U_x + \lambda_2 U^{\frac{1}{2}} + \lambda_3.$$

The functions h and φ arising in (83) satisfy the ODE system

$$\begin{aligned} h_{xx} + \lambda h_x - h^2 + \frac{\lambda_2}{2}h &= 0, \\ \varphi_{xx} + \lambda \varphi_x - h\varphi + \frac{\lambda_2}{2}\varphi + \frac{\lambda_3}{2} &= 0, \end{aligned}$$

which is not integrable. Moreover, there are no any particular solutions of this system in the known books [43, 44]. The trivial solution of the first equation $h = \frac{\lambda_2}{2}$ leads to a particular case of solution (67).

Thus, we have constructed all possible exact solutions, which can be obtained by the application of the Q -conditional symmetry operators arising in theorem 1.

Now we apply the operators arising in theorem 2 to construct exact solutions. First of all we note that cases (i) and (ii) only should be considered because for cases (iv) and (v) the relevant work has been done in the recent paper [32]. Case (iii), of course, cannot produce any new results because the Burgers equation is linearizable by the Cole–Hopf substitution.

Consider case (i) of theorem 2. The operator (16) arising in this case can be successfully applied to construct exact solutions in the explicit form. Omitting rather trivial computations we present the final result: equation

$$U_t = [U^m U_x]_x + \lambda U^{m+1} U_x + \lambda_2 U^{-m}, \quad m \neq -1$$

possesses the solution

$$U = \left[\frac{1}{\lambda t + c_1} \left(-x + \lambda_2(m+1) \left(\lambda \frac{t^2}{2} + c_1 t \right) + c_2 \right) \right]^{\frac{1}{m+1}}, \quad (84)$$

while

$$U = \left[\frac{1}{1 + c_1 e^{-\lambda_1(m+1)t}} \left((m+1) \left(-\frac{\lambda_1}{\lambda} x + \lambda_2 t - \frac{c_1 \lambda_2}{\lambda_1(m+1)} e^{-\lambda_1(m+1)t} \right) + c_2 \right) \right]^{\frac{1}{m+1}} \quad (85)$$

is the exact solution of the nonlinear RDC (15) with $m \neq -1$, $\lambda_1 \neq 0$.

The most cumbersome structure of the conditional symmetry operator occurs in case (ii) of theorem 2. As a consequence, essential difficulties arise if one applies operator (18) for finding exact solutions. On the other hand, it will be shown that many of the exact solutions obtained possess remarkable properties.

Equation (17) and operator (18) by the substitution (36) with $m = -1/2$ are transformed to the forms

$$V_{xx} = V V_t - \lambda V^2 V_x - \frac{\lambda_1^* + 3\lambda V}{3\lambda} \left(\frac{1}{3} \lambda_1^* \lambda V^3 + \lambda_2^* V + \lambda_3^* \right) \tag{86}$$

and

$$Q = \partial_t + (-\lambda V + \lambda_1^*) \partial_x + \left(\frac{1}{3} \lambda_1^* \lambda V^3 + \lambda_2^* V + \lambda_3^* \right) \partial_V, \tag{87}$$

respectively. Hereafter $\lambda_1^* = \frac{3\lambda_1}{2\lambda} \neq 0$, $\lambda_2^* = \frac{\lambda_2}{2}$, $\lambda_3^* = \frac{\lambda_3}{2}$ and $V > 0$ is assumed since (17) contains terms $U^{\frac{1}{2}}$ and $U^{-\frac{1}{2}}$. Instead of the construction of a non-Lie ansatz using operator (87) (in this case it is a cumbersome procedure), one can use the equation $Q(V) = 0$, i.e.

$$V_t = (\lambda V - \lambda_1^*) V_x + \frac{1}{3} \lambda \lambda_1^* V^3 + \lambda_2^* V + \lambda_3^*, \tag{88}$$

to eliminate V_t from (86). In fact, substituting the right-hand side of (88) into (86), one arrives at

$$V_{xx} + \lambda_1^* V V_x + \frac{1}{9} \lambda_1^{*2} V^3 + \frac{1}{3\lambda} \lambda_1^* \lambda_2^* + \frac{1}{3\lambda} \lambda_1^* \lambda_3^* = 0, \tag{89}$$

which is the nonlinear ODE containing variable t as a parameter. Equation (89) is reduced to the form

$$V_{yy} + 3V V_y + V^3 + \frac{3\lambda_2^*}{\lambda_1^* \lambda} V + \frac{3\lambda_3^*}{\lambda_1^* \lambda} = 0 \tag{90}$$

by the simple substitution

$$y = \frac{\lambda_1^*}{3} x. \tag{91}$$

Equation (90) can be transformed into the linear third-order ODE

$$W_{yyy} + 3p W_y + 2q W = 0, \tag{92}$$

where $\frac{\lambda_2^*}{\lambda_1^* \lambda} = p$, $\frac{3\lambda_3^*}{\lambda_1^* \lambda} = 2q$, by the known substitution [43] (see item (6.38))

$$V = \frac{W_y}{W}. \tag{93}$$

According to the classical theory of the linear ODE one needs to solve the algebraic equation

$$k^3 + 3pk + 2q = 0, \tag{94}$$

which corresponds to (92). Hence, four different subcases depending on the values of p and q should be separately considered.

Subcase 1. If $p = q = 0$ then $k_1 = k_2 = k_3 = 0$. The general solution of (92) has the form $W = f + gy + hy^2$ and we arrive at the expression

$$V = \frac{g + 2hy}{f + gy + hy^2}, \tag{95}$$

giving the general solution of the nonlinear ODE (90). Hereafter $f = f(t)$, $g = g(t)$, $h = h(t)$ are arbitrary (at the moment) smooth function and at least one of them must be non-zero. So, (95) with (91) generates the general solution of (89) with $\lambda_1^* \neq 0$. Finally, to obtain the general

solution of system (86) and (88), it is sufficient to substitute (95) with $y = \frac{\lambda_1^*}{3}x$ into the second equation of this system. After the relevant calculations a cumbersome expression is obtained; however, one splits into separate parts for x^n , $n = 0, 1, 2$, and we arrive at the ODE system

$$\begin{aligned}gh_t - g_t h &= \frac{2}{3}\lambda_1^{*2}h^2, \\fh_t - f_t h &= \frac{1}{3}\lambda_1^*h(2\lambda h + \lambda_1^*g), \\fg_t - f_t g &= \frac{1}{3}\lambda_1^*(2\lambda gh - 2\lambda_1^*fh + \lambda_1^*g^2).\end{aligned}\tag{96}$$

System (96) has the similar structure to that from [32] (see formula (60)) and can be solved in a similar way. Substituting the general solution of (96) into (95) and using (91), we find the exact solutions

$$V = \frac{3}{-\lambda_1^{*2}t + \lambda_1^*x + 3c_1}\tag{97}$$

and

$$V = \frac{2\lambda_1^*(x - \lambda_1^*t) + 3c_1}{\frac{\lambda_1^{*2}}{3}(x - \lambda_1^*t)^2 + c_1\lambda_1^*(x - \lambda_1^*t) - 2\lambda\lambda_1^*t + c_2\lambda_1^*}\tag{98}$$

of the equation

$$V_{xx} = VV_t - \lambda V^2V_x - \frac{\lambda_1^*}{3}\lambda V^4 - \frac{\lambda_1^{*2}}{9}V^3.\tag{99}$$

Applying the substitution (36) with $m = -1/2$ to (97)–(99) and renaming the parameters, we arrive at the exact solutions

$$U = \left[\frac{1}{-\frac{3\lambda_1^*}{4\lambda^2}t + \frac{\lambda_1}{2\lambda}x + c_1} \right]^2$$

and

$$U = \left[\frac{2(x - 3\frac{\lambda_1}{2\lambda}t) + \frac{2\lambda c_1}{\lambda_1}}{\frac{\lambda_1}{2\lambda}(x - \frac{3\lambda_1}{2\lambda}t)^2 + c_1(x - \frac{3\lambda_1}{2\lambda}t) - 2\lambda t + c_2} \right]^2$$

of the nonlinear RDC equation

$$U_t = [U^{-\frac{1}{2}}U_x]_x + \lambda U^{\frac{1}{2}}U_x + \lambda_1 U^2 + \frac{\lambda_1^2}{2\lambda}U^{\frac{3}{2}}.$$

Subcase 2. If $p^3 = -q^2 \neq 0$ then $k_1 = \alpha_1 = -2\sqrt[3]{q}$ and $k_2 = k_3 = \alpha_2 = \sqrt[3]{q}$. The general solution of (92) is

$$W = f \exp(\alpha_1 y) + (g + yh) \exp(\alpha_2 y), \quad \alpha_1 = -2\alpha_2$$

so that the expression

$$V = \frac{\alpha_1 f \exp(\alpha_1 y) + (\alpha_2 g + h(\alpha_2 y + 1)) \exp(\alpha_2 y)}{f \exp(\alpha_1 y) + (g + yh) \exp(\alpha_2 y)}\tag{100}$$

presents the general solution of the nonlinear ODE (90). Dealing in a quite similar way to subcase 1, one easily obtains the ODE system

$$\begin{aligned}gh_t - g_t h &= \frac{\lambda_1^*}{3}h^2(2\lambda\alpha_2 + \lambda_1^*), \\fh_t - f_t h &= \lambda_1^*\alpha_2 f h(\lambda\alpha_2 - \lambda_1^*), \\3\alpha_2(fg_t - f_t g) + fh_t - f_t h &= \lambda_1^*\alpha_2 f \left(g \left(\frac{2}{3}\lambda\alpha_2^2 - 3\alpha_2\lambda_1^* - \frac{5}{3} \right) - h(\lambda\alpha_2 + 2\lambda_1^*) \right)\end{aligned}\tag{101}$$

to find the unknown functions $f(t)$, $g(t)$ and $h = h(t)$. It turns out that system (101) has the same structure as that (67) from [32] and its general solution can be constructed. Finally, we find the exact solutions

$$U = \left[\frac{-2c_1\alpha_2 \exp(\beta_0 t - \beta_1 x) + \alpha_2 c_2}{c_1 \exp(\beta_0 t - \beta_1 x) + c_2} \right]^2$$

and

$$U = \left[\frac{-2c_1\alpha_2 \exp(\beta_0 t - \beta_1 x) + c_2\alpha_2(\beta_2 t - \beta_3 x + c_3 + \frac{1}{\alpha_2})}{c_1 \exp(\beta_0 t - \beta_1 x) + c_2(\beta_2 t - \beta_3 x + c_3)} \right]^2$$

of the nonlinear RDC equation

$$U_t = [U^{-\frac{1}{2}} U_x]_x + \lambda U^{\frac{1}{2}} U_x + \left(\lambda_1 U^{\frac{3}{2}} - 3\sqrt{\frac{\lambda_1 \lambda_3^2}{4}} U^{\frac{1}{2}} + \lambda_3 \right) \left(\frac{\lambda_1}{2\lambda^2} + U^{\frac{1}{2}} \right),$$

where $\beta_0 = \frac{3\lambda_1}{2\lambda} \alpha_2 (\frac{3\lambda_1}{2\lambda} - \lambda \alpha_2)$, $\beta_1 = \frac{3\lambda_1}{2\lambda} \alpha_2$, $\beta_2 = -\frac{\lambda_1}{2\lambda} (\frac{3\lambda_1}{2\lambda} + 2\lambda \alpha_2)$, $\beta_3 = -\frac{\lambda_1}{2\lambda}$, $\alpha_2 = \sqrt[3]{\frac{\lambda_3}{2\lambda_1}} \neq 0$.

Subcase 3. If $p^3 + q^2 < 0$ then three roots of (94) are different and real. This case is the most cumbersome because the known Cardano formulae must be used. Let us set $k_1 = \alpha_1$, $k_2 = \alpha_2$ and $k_3 = \alpha_3$, where α_i , $i = 1, 2, 3$, are different real numbers, which are calculated by the Cardano formulae

$$\begin{aligned} \alpha_1 &= -2\sqrt[6]{-p^3} \cos \left(\frac{1}{3} \arctan \left(\frac{\sqrt{-p^3 - q^2}}{q} \right) \right), \\ \alpha_2 &= 2\sqrt[6]{-p^3} \cos \left(\frac{1}{3} \arctan \left(\frac{\sqrt{-p^3 - q^2}}{q} \right) - \frac{\pi}{3} \right), \\ \alpha_3 &= 2\sqrt[6]{-p^3} \cos \left(\frac{1}{3} \arctan \left(\frac{\sqrt{-p^3 - q^2}}{q} \right) + \frac{\pi}{3} \right), \end{aligned} \tag{102}$$

if $q > 0$, by the formulae

$$\begin{aligned} \alpha_1 &= 2\sqrt[6]{-p^3} \cos \left(\frac{1}{3} \arctan \left(\frac{\sqrt{-p^3 - q^2}}{q} \right) \right), \\ \alpha_2 &= -2\sqrt[6]{-p^3} \cos \left(\frac{1}{3} \arctan \left(\frac{\sqrt{-p^3 - q^2}}{q} \right) - \frac{\pi}{3} \right), \\ \alpha_3 &= -2\sqrt[6]{-p^3} \cos \left(\frac{1}{3} \arctan \left(\frac{\sqrt{-p^3 - q^2}}{q} \right) + \frac{\pi}{3} \right), \end{aligned} \tag{103}$$

if $q < 0$ and by the formulae

$$\alpha_1 = 0, \quad \alpha_2 = \sqrt{-3p}, \quad \alpha_3 = -\sqrt{-3p}, \tag{104}$$

if $q = 0$. The general solution of (92) is

$$W = f \exp(\alpha_1 y) + g \exp(\alpha_2 y) + h \exp(\alpha_3 y),$$

and it leads to the general solution

$$V = \frac{\alpha_1 f \exp(\alpha_1 y) + \alpha_2 g \exp(\alpha_2 y) + \alpha_3 h \exp(\alpha_3 y)}{f \exp(\alpha_1 y) + g \exp(\alpha_2 y) + h \exp(\alpha_3 y)} \tag{105}$$

of the nonlinear ODE (90).

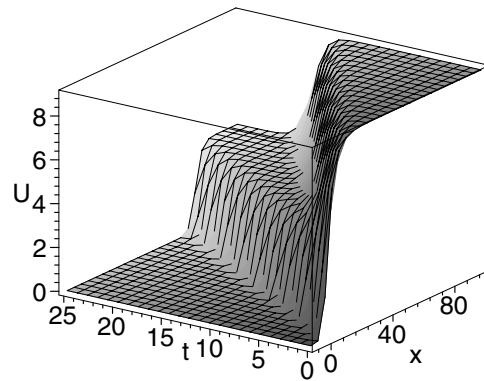


Figure 5. Exact solution (107) with $\lambda = 1$, $\lambda_1 = 0.5$, $c_1 = c_2 = c_3 = 1$, $\alpha_1 = 0.1$, $\alpha_2 = 2$, $\alpha_3 = 3$.

Substituting (105) with $y = \frac{\lambda_1^*}{3}x$ into (88) and conducting the relevant calculations and splits we again arrive at the ODE system to find the unknown functions $f(t)$, $g(t)$ and $h(t)$. This system has the form

$$\begin{aligned} f_t g - f g_t &= -\frac{\lambda_1^*}{3} f g (\lambda(\alpha_1^2 - \alpha_2^2) + \lambda_2(\alpha_1 - \alpha_2)), \\ f_t h - f h_t &= -\frac{\lambda_1^*}{3} f h (\lambda(\alpha_1^2 - \alpha_3^2) + \lambda_2(\alpha_1 - \alpha_3)), \\ g_t h - g h_t &= -\frac{\lambda_1^*}{3} g h (\lambda(\alpha_2^2 - \alpha_3^2) + \lambda_2(\alpha_2 - \alpha_3)) \end{aligned} \quad (106)$$

and is fully integrable [32]. Its general solution leads to the exact solution

$$U = \left[\frac{\alpha_1 c_1 \exp(\beta_1 t + \gamma_1 x) + \alpha_2 c_2 \exp(\beta_2 t + \gamma_2 x) + \alpha_3 c_3 \exp(\beta_3 t + \gamma_3 x)}{c_1 \exp(\beta_1 t + \gamma_1 x) + c_2 \exp(\beta_2 t + \gamma_2 x) + c_3 \exp(\beta_3 t + \gamma_3 x)} \right]^2 \quad (107)$$

of the nonlinear RDC equation (17) with $\lambda_1 \neq 0$. Here, $\beta_i = -\frac{\lambda_1}{2\lambda} (\lambda \alpha_i^2 + \frac{3\lambda_1}{2\lambda} \alpha_i)$, $\gamma_i = \frac{\lambda_1}{2\lambda} \alpha_i$, $i = 1, 2, 3$, and the roots α_i , $i = 1, 2, 3$, are determined by formulae (102)–(104) depending on the q sign. In the case $c_i > 0$, $i = 1, 2, 3$, this type of exact solutions is known in applications as two-shock waves (see, e.g., [26]). Note that such solutions satisfy the zero Neumann conditions, $U_x = 0$ at $x = \pm\infty$. An example of such a solution is presented in figure 5.

Consider the generalized FN equation with the fast diffusion (35). In the case $\lambda_2 \neq 0$ and $\lambda_3 = 0$, we immediately obtain $p < 0$ and $q = 0$, therefore formulae (104) and (107) give the solution

$$U = \left[\frac{c_2 \exp\left(\frac{\lambda_2}{2\lambda} \left(\left(\lambda - \frac{3\lambda_2}{2\lambda}\right)t - x\right)\right) - c_3 \exp\left(\frac{\lambda_2}{2\lambda} \left(\left(\lambda + \frac{3\lambda_2}{2\lambda}\right)t + x\right)\right)}{c_1 + c_2 \exp\left(\frac{\lambda_2}{2\lambda} \left(\left(\lambda - \frac{3\lambda_2}{2\lambda}\right)t - x\right)\right) + c_3 \exp\left(\frac{\lambda_2}{2\lambda} \left(\left(\lambda + \frac{3\lambda_2}{2\lambda}\right)t + x\right)\right)} \right]^2. \quad (108)$$

Taking into account formula (36) with $m = -1/2$, we note that solution (108) with arbitrary c_1 , c_2 and c_3 is not valid in the domain $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$. However, setting, for example, $c_3 = 0$ and positive constants c_1, c_2 , we obtain traveling wave solution, which is valid in this domain. An example of such a solution is presented in figure 6. It should be stressed that similar solutions also possess the classical FN equation [45] and the generalized FN equation (31) [32].

Subcase 4. If $p^3 + q^2 > 0$ then three roots of (94) are different and two of them are complex conjugate. The Cardano formulae should be again applied. Setting $k_1 = \alpha$, $k_{2,3} = a \pm ib$,

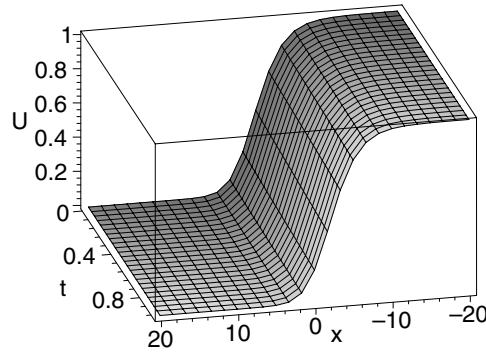


Figure 6. Exact solution (108) with $\lambda = 1, \lambda_2 = 1, c_1 = 1, c_2 = 1, c_3 = 0, \alpha_2 = 1, \alpha_3 = -1$.

where

$$\begin{aligned} \alpha &= \sqrt[3]{-q + \sqrt{p^3 + q^2}} - \sqrt[3]{q + \sqrt{p^3 + q^2}}, \\ a &= -\frac{1}{2}(\sqrt[3]{-q + \sqrt{p^3 + q^2}} - \sqrt[3]{q + \sqrt{p^3 + q^2}}), \\ b &= \frac{\sqrt{3}}{2}(\sqrt[3]{-q + \sqrt{p^3 + q^2}} + \sqrt[3]{q + \sqrt{p^3 + q^2}}), \end{aligned} \tag{109}$$

the general solution of (92) may be presented in the form

$$W = f \exp(\alpha y) + (g \cos(by) + h \sin(by)) \exp(\alpha y), \quad \alpha = -2a.$$

Using (93) one arrives at the general solution

$$V = \frac{\alpha f \exp(\alpha y) + (g(a \cos(by) - b \sin(by)) + h(b \cos(by) + a \sin(by))) \exp(\alpha y)}{f \exp(\alpha y) + (g \cos(by) + h \sin(by)) \exp(\alpha y)} \tag{110}$$

of the nonlinear ODE (90). The analog of (106) in this case takes the form

$$\begin{aligned} -3a(f_t g - f g_t) + b(fh_t - f_t h) &= bfh(\lambda_2^* - 2\lambda_1^{*2}a) + \frac{\lambda_1^*}{3}\lambda a f g(2a^2 + 2b^2 + 5a^3 + ab^2), \\ -3a(f_t h - f h_t) + b(f_t g - f g_t) &= -bfg(\lambda_2^* - 2\lambda_1^{*2}a) \\ &+ \frac{\lambda_1^*}{3}\lambda a f h(2a^2 + 2b^2 + 5a^3 + ab^2), \\ gh_t - g_t h &= \frac{\lambda_1^*}{3}b(2\lambda a + \lambda_2)(g^2 + h^2). \end{aligned} \tag{111}$$

It should be stressed that the ODE system (111) has essentially different structures from those presented above and its solving takes a lot of efforts. We were able to realize all necessary computations, which are omitting here, and to check the result using the program package MATHEMATICA 5.0. Finally, the exact solution

$$U = \left[\frac{-2c_1 a \exp(\beta_0 x + \beta_1 t) + c_2(a \cos(\beta_2 x + \beta_3 t - c_3) - b \sin(\beta_2 x + \beta_3 t - c_3))}{c_1 \exp(\beta_0 x + \beta_1 t) + c_2 \cos(\beta_2 x + \beta_3 t - c_3)} \right]^2 \tag{112}$$

of the nonlinear RDC equation (17) with $\lambda_1 \neq 0$ has been found. Here $\beta_0 = -\frac{3\lambda_1}{2\lambda}a, \beta_1 = -\frac{\lambda_1}{2\lambda}(\lambda(b^2 + 3a^2) - \frac{9\lambda_1}{2\lambda}a), \beta_2 = \frac{\lambda_1}{2\lambda}b, \beta_3 = -\frac{\lambda_1}{2\lambda}b(2\lambda a + \frac{3\lambda_1}{2\lambda})$, and a and b are determined by formulae (109). Note that quasi-periodic periodic solutions of the similar form were also obtained for the reaction–diffusion equation (32) with $\lambda = 0$ [29, 33].

5. Conclusions

In this paper, theorems 1 and 2 giving a complete description of Q -conditional symmetries of the nonlinear RDC equations (2) and (3) are proved. It should be stressed that all the Q -conditional symmetry operators listed in theorems 1 and 2 contain the same nonlinearities with respect to the dependent variable U as the relevant RDC equations. Analogous results were earlier obtained for single reaction–diffusion equations [23, 27–29].

However, we note that there is the essential difference between the RDC equations (2) and (3) and the relevant RD equation

$$U_t = [U^m U_x]_x + C(U).$$

For example, the Murray-type equation (22) admits the Q -conditional symmetry (23), while the RD equation with this term, i.e. the Fisher-type equation

$$U_t = U_{xx} + \lambda_0 + \lambda_1 U + \lambda_2 U^2, \quad \lambda_2 \neq 0,$$

does not possess one. Similarly, the RDC equation (17) possessing the Q -conditional symmetry (18) does not have an analog among reaction–diffusion equations with the diffusivity $U^{-\frac{1}{2}}$.

The RDC equations listed in theorems 1 and 2 contain several well-known equations arising in applications and their direct generalizations. In the particular case, the Murray equation (27), its porous analog (26) and its analog (28) with the fast diffusion (see also (29)–(30)); the Fitzhugh–Nagumo equation [37] with the convective term

$$U_t = U_{xx} + \lambda U U_x + \lambda_3 U (U - \delta)(1 - U), \quad 0 < \delta < 1 \quad (113)$$

and its analog (35) with the fast diffusion; the Kolmogorov–Petrovskii–Piskunov equation [38] with the convective term

$$U_t = U_{xx} + \lambda U U_x + \lambda_3 U (1 - U)^2 \quad (114)$$

and the Newell–Whitehead equation with the convective term

$$U_t = U_{xx} + \lambda U U_x + \lambda_3 U^3 - \lambda_1 U. \quad (115)$$

A further generalization of the RDC equations (2) and (3) reads

$$U_t = [U^m U_x]_x + \lambda U^n U_x + C(U), \quad \lambda \neq 0, \quad (116)$$

where m and n are the arbitrary constants. The work is in progress on the complete description of Q -conditional symmetry of (116) and the RDC equation with exponential nonlinearities

$$U_t = [\exp(mU)U_x]_x + \lambda \exp(nU)U_x + C(U), \quad \lambda \neq 0.$$

It is well known that new Q -conditional symmetries and ad hoc methods do not guarantee the construction of exact solutions, which cannot be obtained by the Lie machinery (see non-trivial examples in [41, 42]). In this paper, several exact solutions were constructed using the conditional symmetries arising in theorems 1 and 2. It was shown that these solutions are not obtainable by Lie symmetries; however, they contain the known plane wave solutions as particular cases. Many of the solutions obtained possess attractive properties and can be used for further investigation of the relevant boundary-value problems. In the particular case, we established that the zero Dirichlet and Neumann conditions, i.e. typical boundary conditions for mathematical models arising in physics and biology, can be satisfied by the relevant fitting of constants c_1 , c_2 and c_3 (see the solutions presented in figures 1, 2 and 4–6).

To the best of our knowledge many of the solutions presented above are new. However, we noted that some of them can be derived from the recent paper [46]. In fact, if one applies substitution (36) to the RDC equation (72) and its solutions (69)–(71), then equation (54)

[46] with (65)–(66) [46] and $\alpha(s) = s^n$, and solutions (69)–(71) [46] are exactly obtained. Nevertheless, the authors of that paper do not use any symmetries to construct exact solutions, formula (72) [46] is nothing else but the equation $Q(V) = 0$, where Q is the conditional symmetry operator (47) of (46). Thus, we obtain new confirmation of the known idea (see, i.e., [12, 21, 23]) that any exact solution can be obtained by the relevant Lie or conditional symmetry operator.

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